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Two extragradient methods for generalized mixed equilibrium problems, nonexpansive mappings and monotone mappings☆

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ABSTRACT

In this paper, we introduce iterative schemes based on the extragradient method for finding a common element of the set of solutions of a generalized mixed equilibrium problem and the set of fixed points of an infinite (a finite) family of nonexpansive mappings and the set of solutions of a variational inequality problem for a monotone, Lipschitz continuous mapping. We obtain some weak convergence theorems for the sequences generated by these processes in Hilbert spaces. The results in this paper generalize, extend and unify some well-known weak convergence theorems in the literature.

1. Introduction

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$ and let C be a nonempty closed convex subset of H. Let $B: C \to H$ be a nonlinear mapping and let $\varphi: C \to R \cup \{+\infty\}$ be a function and F be a bifunction from $C \times C$ to R, where R is the set of real numbers. Peng and Yao [1] considered the following generalized mixed equilibrium problem:

$$\text{Finding } x \in C \text{ such that } F(x, y) + \varphi(y) + \langle Bx, y - x \rangle \geq \varphi(x), \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $GMEP(F, \varphi, B)$. It is easy to see that x is a solution of problem (1.1) implies that $x \in \operatorname{dom} \varphi = \{x \in C \mid \varphi(x) < +\infty\}$.

If $B = 0$, then the generalized mixed equilibrium problem (1.1) becomes the following mixed equilibrium problem:

$$\text{Finding } x \in C \text{ such that } F(x, y) + \varphi(y) \geq \varphi(x), \quad \forall y \in C. \tag{1.2}$$

Problem (1.2) was studied by Ceng and Yao [2] and Peng and Yao [3,4]. The set of solutions of (1.2) is denoted by $MEP(F, \varphi)$.

If $\varphi = 0$, then the generalized mixed equilibrium problem (1.1) becomes the following generalized equilibrium problem:

$$\text{Finding } x \in C \text{ such that } F(x, y) + \langle Bx, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

Problem (1.3) was studied by Takahashi and Takahashi [5]. The set of solutions of (1.3) is denoted by $GEP(F, B)$.

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If $\varphi = 0$ and $B = 0$, then the generalized mixed equilibrium problem (1.1) becomes the following equilibrium problem:

$$\text{Finding } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions of (1.4) is denoted by $EP(F)$.

If $F(x, y) = 0$ for all $x, y \in C$, the generalized mixed equilibrium problem (1.1) becomes the following generalized variational inequality problem:

$$\text{Finding } x \in C \text{ such that } \varphi(y) + \langle Bx, y - x\rangle \geq \varphi(x), \quad \forall y \in C. \tag{1.5}$$

The set of solutions of (1.5) is denoted by $GVI(C, \varphi, B)$.

If $\varphi = 0$ and $F(x, y) = 0$ for all $x, y \in C$, the generalized mixed equilibrium problem (1.1) becomes the following variational inequality problem:

$$\text{Finding } x \in C \text{ such that } \langle Bx, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.6}$$

The set of solutions of (1.6) is denoted by $VI(C, B)$.

If $B = 0$ and $F(x, y) = 0$ for all $x, y \in C$, the generalized mixed equilibrium problem (1.1) becomes the following minimization problem:

$$\text{Finding } x \in C \text{ such that } \varphi(y) \geq \varphi(x), \quad \forall y \in C. \tag{1.7}$$

The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games and others; see for instance, [1–7].

Peng and Yao [1] introduced an iterative scheme for finding a common element of the set of solutions of problem (1.1), the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality for a monotone, Lipschitz continuous mapping and obtain a strong convergence theorem. Ceng and Yao [2] introduced an iterative scheme for finding a common element of the set of solutions of problem (1.2) and the set of common fixed points of a family of finitely nonexpansive mappings in a Hilbert space and obtained a strong convergence theorem. Peng and Yao [3] introduced an iterative scheme for finding a common element of the set of solutions of problem (1.2), the set of fixed points of a family of finitely nonexpansive mappings and the set of solutions of a variational inequality for a monotone, Lipschitz continuous mapping and obtain a strong convergence theorem. Peng and Yao [4] introduced an approximation scheme combining the viscosity method with parallel method for finding a common element of the set of solutions of a generalized equilibrium problem and the set of fixed points of a family of finitely strict pseudocontractions. Takahashi and Takahashi [5] introduced an iterative scheme for finding a common element of the set of solutions of problem (1.3) and the set of fixed points of a nonexpansive mapping in a Hilbert space and proved a strong convergence theorem.

Some methods have been proposed to solve problem (1.4); see, for instance, [6–13] and the references therein. Recently, Combettes and Hirstoaga [8] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem. Takahashi and Takahashi [9] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of problem (1.4) and the set of fixed points of a nonexpansive mapping and proved a strong convergence theorem in a Hilbert space. Su, Shang and Qin [10] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of problem (1.4) and the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem for an α-inverse strongly monotone mapping in a Hilbert space and obtain a strong convergence theorem. Tada and Takahashi [11] introduced some iterative schemes for finding a common element of the set of solutions of problem (1.4) and the set of fixed points of a nonexpansive mapping in a Hilbert space and obtained both strong convergence theorem and weak convergence theorem.

On the other hand, for solving the variational inequality problem in the finite-dimensional Euclidean spaces, Korpelevich [14] introduced the following so-called extragradient method:

$$\begin{cases} x_1 = x \in C \\ y_n = P_C(x_n - \lambda Bx_n), \\ x_{n+1} = P_C(x_n - \lambda By_n), \end{cases} \tag{1.8}$$

for every $n = 0, 1, 2, \ldots$, where $\lambda \in (0, \frac{1}{K})$, C is a closed convex subset from R^n, $B : C \to R^n$ is a monotone and K-Lipschitz continuous mapping and P_C is the metric projection of R^n into C. She showed that if $VI(C, B)$ is nonempty, then the sequences $\{x_n\}$ and $\{y_n\}$, generated by (1.8), converge to the same point $z \in VI(C, A)$. The idea of the extragradient iterative process introduced by Korpelevich was successfully generalized and extended not only in Euclidean but also in Hilbert and Banach spaces; see, e.g., the recent papers of He, Yang and Yuan [15], Gárciga Otero and Iuzem [16], Solodov and Svaiter [17], Solodov [18]. Moreover, Zeng and Yao [19] and Nadezhkina and Takahashi [20] introduced some iterative processes based on the extragradient method for finding the common element of the set of fixed points of nonexpansive mappings and the set of solutions of a variational inequality problem for a monotone, Lipschitz continuous mapping. Yao and Yao [21] introduced an iterative process based on the extragradient method for finding the common element of the set of fixed points of nonexpansive mappings and the set of solutions of a variational inequality problem for an α-inverse strongly monotone mapping. Plubtieng and Punpaeng [13] introduced an iterative process based on the extragradient method for finding the common element of the set of fixed points of nonexpansive mappings, the set of solutions of an equilibrium problem and the set of solutions of a variational inequality problem for α-inverse strongly monotone mappings.

In the present paper, we introduce some iterative schemes based on the extragradient method for finding a common element of the set of solutions of a generalized mixed equilibrium problem, the set of fixed points of an infinite (a finite) family of nonexpansive mappings and the set of solutions of a variational inequality problem for a monotone, Lipschitz continuous mapping. We obtain some weak convergence theorems for the sequences generated by these processes. The results in this paper generalize, extend and unify some well-known weak convergence theorems in the literature.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. Let symbols $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively. In a real Hilbert space H, it is well known that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.1}$$

for all $x, y \in H$ and $\lambda \in [0, 1]$. It is also known that H satisfies the Opial condition [22], i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $x \neq y$.

For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| \le \|x - y\|$ for all $y \in C$. The mapping P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping from H onto C. It is also known that $P_C(x) \in C$ and

$$\langle x - P_C(x), P_C(x) - y \rangle \ge 0 \tag{2.2}$$

for all $x \in H$ and $y \in C$.

It is easy to see that (2.2) is equivalent to

$$\|x - y\|^2 \ge \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2 \tag{2.3}$$

for all $x \in H$ and $y \in C$.

A mapping A of C into H is called monotone if

$$\langle Ax - Ay, x - y \rangle \ge 0$$

for all $x, y \in C$. A mapping A of C into H is called inverse strongly monotone with a modulus α (in short, α-inverse strongly monotone) if there exists a positive real number α such that

$$\langle x - y, Ax - Ay \rangle \ge \alpha\|Ax - Ay\|^2$$

for all $x, y \in C$. A mapping $A : C \to H$ is called Lipschitz continuous with Lipschitz constant K (in short, K-Lipschitz continuous) if there exists a positive real number K such that

$$\|Ax - Ay\| \le K\|x - y\|$$

for all $x, y \in C$. Recall that a mapping S of C into itself is nonexpansive [23] if

$$\|Sx - Sy\| \le \|x - y\| \quad \text{for all } x, y \in C.$$

A mapping T of a closed convex subset C into itself is pseudocontractive if

$$\langle Tx - Ty, x - y \rangle \le \|x - y\|^2$$

for all $x, y \in C$; see [24]. Obviously, the class of pseudocontractive mappings is more general than the class of nonexpansive mappings.

We denote the set of fixed points of S by $Fix(S)$. It is easy to see that if A is inverse strongly monotone, then A is monotone and Lipschitz continuous. The converse is not true in general. The class of inverse strongly monotone mappings does not contain some important classes of mappings even in a finite-dimensional case. For example, if the corresponding matrix of a linear complementarity problem is positively semidefinite, but not positively definite, then the mapping A will be monotone and Lipschitz continuous, but not inverse strongly monotone.

Let A be a monotone mapping from C into H. In the context of the variational inequality problem the characterization of projection (2.2) implies the following:

$$u \in VI(C, A) \Rightarrow u = P_C(u - \lambda Au), \quad \lambda > 0,$$

and

$$u = P_C(u - \lambda Au) \quad \text{for some } \lambda > 0 \Rightarrow u \in VI(C, A).$$

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H, f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph of $G(T)$ is not properly contained in the graph of any other monotone mapping. In other words a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be a monotone, K-Lipschitz continuous mapping of C into H and let $N_C v$ be normal cone to C at $v \in C$, i.e, $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$ (see [25] and [26]).

For solving the generalized mixed equilibrium problem and the mixed equilibrium problem, let us give the following assumptions for the bifunction F, the function φ and the set C:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e. $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;
(A3) for each $y \in C, x \mapsto F(x, y)$ is weakly upper semicontinuous;
(A4) for each $x \in C, y \mapsto F(x, y)$ is convex;
(A5) for each $x \in C, y \mapsto F(x, y)$ is lower semicontinuous;
(B1) For each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C \cap \operatorname{dom}(\varphi)$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z);$$

(B2) C is a bounded set.

Lemma 2.1 (*[1,3,4]*). *Let C be a nonempty closed convex subset of a Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)–(A5) *and let $\varphi : C \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either* (B1) *or* (B2) *holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z, y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \geq \varphi(z), \forall y \in C \right\}$$

for all $x \in H$. Then the following conclusions hold:

(1) *For each $x \in H$, $T_r(x) \neq \emptyset$;*
(2) *T_r is single-valued;*
(3) *T_r is firmly nonexpansive, i.e, for any $x, y \in H$,*

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle;$$

(4) *$Fix(T_r) = MEP(F, \varphi)$;*
(5) *$MEP(F, \varphi)$ is closed and convex.*

Lemma 2.2 (*[27]*). *Let H be a real Hilbert space, let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < a \leq \alpha_n \leq b < 1$ for all $n = 0, 1, 2, \ldots$, and let $\{v_n\}$ and $\{w_n\}$ be sequences in H such that* $\limsup_{n\to\infty} \|v_n\| \leq c$, $\limsup_{n\to\infty} \|w_n\| \leq c$, $\limsup_{n\to\infty} \|\alpha_n v_n + (1 - \alpha_n)w_n\| = c$ *for some $c \geq 0$. Then,* $\lim_{n\to\infty} \|v_n - w_n\| = 0$.

Lemma 2.3 (*[28]*). *Let H be a real Hilbert space, let D be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H. Suppose that, for all $u \in D$,*

$$\|x_{n+1} - u\| \leq \|x_n - u\|,$$

for every $n = 0, 1, 2, \ldots$. Then, the sequence $\{P_D x_n\}$ converges strongly to some $z \in D$.

3. The case of an infinite family of nonexpansive mappings

In this section, we show a weak convergence theorem of an iterative algorithm based on extragradient method which solves the problem of finding a common element of the set of solutions of a generalized mixed equilibrium problem, the set of fixed points of an infinite family of nonexpansive mappings and the set of solutions of a variational inequality problem for a monotone, Lipschitz continuous mapping in a Hilbert space.

Let $S_1, S_2, \ldots$ be an infinite family of nonexpansive mappings of C into itself and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 \leq \xi_i \leq 1$ for every $i \in N$. For any $n \in N$, define mapping W_n of C into C as follows:

$$U_{n,n+1} = I,$$
$$U_{n,n} = \xi_n S_n U_{n,n+1} + (1-\xi_n)I,$$
$$U_{n,n-1} = \xi_{n-1} S_{n-1} U_{n,n} + (1-\xi_{n-1})I,$$
$$\dots$$
$$U_{n,k} = \xi_k S_k U_{n,k+1} + (1-\xi_k)I,$$
$$U_{n,k-1} = \xi_{k-1} S_{k-1} U_{n,k} + (1-\xi_{k-1})I,$$
$$\dots$$
$$U_{n,2} = \xi_2 S_2 U_{n,3} + (1-\xi_2)I,$$
$$W_n = U_{n,1} = \xi_1 S_1 U_{n,2} + (1-\xi_1)I.$$

Such a mapping W_n is called the W-mapping generated by $S_n, S_{n-1}, \dots, S_1$ and $\xi_n, \xi_{n-1}, \dots, \xi_1$; see [29].

Lemma 3.1 (*[29]*). *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $S_1, S_2, \dots$ be an infinite family of nonexpansive mappings of C into itself such that $\cap_{i=1}^{\infty} Fix(S_i)$ is nonempty, and let $\xi_1, \xi_2, \dots$ be real numbers such that $0 < \xi_i \le d < 1$ for every $i \in N$. Then for every $x \in C$ and $k \in N$, $\lim_{n\to\infty} U_{n,k}x$ exists.*

Remark 3.1. Using Lemma 3.1, one can define a mappings $U_{\infty,k}$ and W of C into itself as follows:

$$U_{\infty,k}x = lim_{n\to\infty} U_{n,k}x$$

and $Wx = lim_{n\to\infty} W_n x = lim_{n\to\infty} U_{n,1}x$ for every $x \in C$. Such a mapping W is called the W-mapping generated by $S_1, S_2, \dots$ and $\xi_1, \xi_2, \dots$. Since W_n is nonexpansive, $W : C \to C$ is also nonexpansive. Indeed, observe that for each $x, y \in C$

$$\|Wx - Wy\| = lim_{n\to\infty} \|W_n x - W_n y\| \le \|x - y\|.$$

If $\{x_n\}$ is a bounded sequence in C, then we have

$$lim_{n\to\infty} \|Wx_n - W_n x_n\| = 0.$$

Lemma 3.2 (*[29]*). *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $S_1, S_2, \dots$ be an infinite family of nonexpansive mappings of C into itself such that $\cap_{i=1}^{\infty} Fix(S_i)$ is nonempty, and let $\xi_1, \xi_2, \dots$ be real numbers such that $0 < \xi_i \le d < 1$ for every $i \in N$. Then $Fix(W) = \cap_{i=1}^{\infty} Fix(S_i)$.*

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)–(A5) *and $\varphi : C \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be a monotone and K-Lipschitz continuous mapping from C into H and B be an α-inverse strongly monotone mapping from C into H. Let $S_1, S_2, \dots$ be a family of infinitely nonexpansive mappings of C into itself such that $\Omega = \cap_{i=1}^{\infty} Fix(S_i) \cap VI(C, A) \cap GMEP(F, \varphi, B) \ne \emptyset$ and let $\xi_1, \xi_2, \dots$ be real numbers such that $0 < \xi_i \le \delta < 1$ for every $i \in N$. For any $n \in N$, let W_n be the W-mapping from C into itself generated by $S_n, S_{n-1}, \dots, S_1$ and $\xi_n, \xi_{n-1}, \dots, \xi_1$. Assume that either* (B1) *or* (B2) *holds. Let $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) W_n P_C(u_n - \lambda_n A y_n), \end{cases}$$

for every $n = 1, 2, \dots$. If $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{K})$, $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$ and $\{r_n\} \subset [\gamma, \tau]$ for some $\gamma, \tau \in (0, 2\alpha)$, then, $\{x_n\}$ converges weakly to $w \in \Omega$, where $w = \lim_{n\to\infty} P_{\Omega}(x_n)$.

Proof. Put $t_n = P_C(u_n - \lambda_n A y_n)$ for every $n = 1, 2, \dots$. Let $u \in \Omega$ and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.1. Then $u = P_C(u - \lambda_n Au) = T_{r_n}(u - r_n Bu)$. From $u_n = T_{r_n}(x_n - r_n Bx_n) \in C$ and the α-inverse strongly monotonicity of B, we have

$$\begin{aligned} \|u_n - u\|^2 &= \|T_{r_n}(x_n - r_n Bx_n) - T_{r_n}(u - r_n Bu)\|^2 \\ &\le \|x_n - r_n Bx_n - (u - r_n Bu)\|^2 \\ &\le \|x_n - u\|^2 - 2r_n\langle x_n - u, Bx_n - Bu\rangle + r_n^2\|Bx_n - Bu\|^2 \\ &\le \|x_n - u\|^2 - 2r_n\alpha\|Bx_n - Bu\|^2 + r_n^2\|Bx_n - Bu\|^2 \\ &= \|x_n - u\|^2 + r_n(r_n - 2\alpha)\|Bx_n - Bu\|^2 \\ &\le \|x_n - u\|^2. \end{aligned} \tag{3.1}$$

From (2.3), the monotonicity of A, and $u \in VI(C, A)$, we have

$$\begin{aligned}\|t_n - u\|^2 &\le \|u_n - \lambda_n Ay_n - u\|^2 - \|u_n - \lambda_n Ay_n - t_n\|^2 \\ &= \|u_n - u\|^2 - \|u_n - t_n\|^2 + 2\lambda_n \langle Ay_n, u - t_n\rangle \\ &= \|u_n - u\|^2 - \|u_n - t_n\|^2 + 2\lambda_n(\langle Ay_n - Au, u - y_n\rangle + \langle Au, u - y_n\rangle + \langle Ay_n, y_n - t_n\rangle) \\ &\le \|u_n - u\|^2 - \|u_n - t_n\|^2 + 2\lambda_n \langle Ay_n, y_n - t_n\rangle \\ &\le \|u_n - u\|^2 - \|u_n - y_n\|^2 - 2\langle u_n - y_n, y_n - t_n\rangle - \|y_n - t_n\|^2 + 2\lambda_n\langle Ay_n, y_n - t_n\rangle \\ &= \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle u_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle.\end{aligned}$$

Further, since $y_n = P_C(u_n - \lambda_n Au_n)$ and A is K-Lipschitz continuous, we get

$$\begin{aligned}\langle u_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle &= \langle u_n - \lambda_n Au_n - y_n, t_n - y_n\rangle + \langle \lambda_n Au_n - \lambda_n Ay_n, t_n - y_n\rangle \\ &\le \langle \lambda_n Au_n - \lambda_n Ay_n, t_n - y_n\rangle \\ &\le \lambda_n K\|u_n - y_n\|\|t_n - y_n\|.\end{aligned}$$

Thus,

$$\begin{aligned}\|t_n - u\|^2 &\le \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n K\|u_n - y_n\|\|t_n - y_n\| \\ &\le \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + {\lambda_n}^2K^2\|u_n - y_n\|^2 + \|t_n - y_n\|^2 \\ &= \|u_n - u\|^2 + ({\lambda_n}^2K^2 - 1)\|u_n - y_n\|^2. \\ &\le \|u_n - u\|^2. \end{aligned} \tag{3.2}$$

By Lemma 3.1 in [29], we know that W_n is nonexpansive and $F(W_n) = \cap_{i=1}^n Fix(S_i)$. It follows from (2.1), (3.1) and (3.2), $x_{n+1} = \alpha_n x_n + (1 - \alpha_n)W_n t_n$ and $u = W_n u$ that

$$\begin{aligned}\|x_{n+1} - u\|^2 &= \|\alpha_n x_n + (1 - \alpha_n)W_n t_n - u\|^2 \\ &= \alpha_n\|x_n - u\|^2 + (1 - \alpha_n)\|W_n t_n - u\|^2 - \alpha_n(1 - \alpha_n)\|x_n - W_n t_n\|^2 \\ &\le \alpha_n\|x_n - u\|^2 + (1 - \alpha_n)\|W_n t_n - u\|^2 \\ &\le \alpha_n\|x_n - u\|^2 + (1 - \alpha_n)\|t_n - u\|^2 \\ &\le \alpha_n\|x_n - u\|^2 + (1 - \alpha_n)[\|u_n - u\|^2 + ({\lambda_n}^2K^2 - 1)\|u_n - y_n\|^2] \\ &\le \|x_n - u\|^2 + (1 - \alpha_n)({\lambda_n}^2K^2 - 1)\|u_n - y_n\|^2 \\ &\le \|x_n - u\|^2, \end{aligned} \tag{3.3}$$

for every $n = 1, 2, \ldots$. Therefore, there exists $\theta = \lim_{n\to\infty} \|x_n - u\|$ and $\{x_n\}$ is bounded. From (3.1) and (3.2), we also obtain that $\{t_n\}$ and $\{u_n\}$ are bounded.

By (3.3), we have

$$\|u_n - y_n\|^2 \le \frac{1}{(1 - \alpha_n)(1 - {\lambda_n}^2K^2)}\left(\|x_n - u\|^2 - \|x_{n+1} - u\|^2\right). \tag{3.4}$$

Hence, $\|u_n - y_n\| \to 0$.

By the same process as in (3.2), we also have

$$\begin{aligned}\|t_n - u\|^2 &\le \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n K\|u_n - y_n\|\|t_n - y_n\| \\ &\le \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + \|u_n - y_n\|^2 + {\lambda_n}^2K^2\|t_n - y_n\|^2 \\ &= \|u_n - u\|^2 + ({\lambda_n}^2K^2 - 1)\|y_n - t_n\|^2.\end{aligned}$$

Then, proceeding similarly (3.3), we have

$$\begin{aligned}\|x_{n+1} - u\|^2 &\le \alpha_n\|x_n - u\|^2 + (1 - \alpha_n)\|t_n - u\|^2 \\ &\le \alpha_n\|x_n - u\|^2 + (1 - \alpha_n)[\|u_n - u\|^2 + ({\lambda_n}^2K^2 - 1)\|y_n - t_n\|^2] \\ &\le \|x_n - u\|^2 + (1 - \alpha_n)({\lambda_n}^2K^2 - 1)\|y_n - t_n\|^2 \\ &\le \|x_n - u\|^2,\end{aligned}$$

from which it follows that

$$\|t_n - y_n\|^2 \le \frac{1}{(1 - \alpha_n)(1 - {\lambda_n}^2K^2)}\left(\|x_n - u\|^2 - \|x_{n+1} - u\|^2\right).$$

Hence, $\|t_n - y_n\| \to 0$. From $\|u_n - t_n\| \le \|u_n - y_n\| + \|y_n - t_n\|$ we also have $\|u_n - t_n\| \to 0$. As A is K-Lipschitz continuous, we have $\|Ay_n - At_n\| \to 0$.

By (3.3) and (3.1), we have

$$\begin{aligned}\|x_{n+1}-u\|^2 &\le \alpha_n\|x_n-u\|^2+(1-\alpha_n)[\|u_n-u\|^2+(\lambda_n^2K^2-1)\|u_n-y_n\|^2]\\ &\le \alpha_n\|x_n-u\|^2+(1-\alpha_n)\|u_n-u\|^2\\ &\le \alpha_n\|x_n-u\|^2+(1-\alpha_n)[\|x_n-u\|^2+r_n(r_n-2\alpha)\|Bx_n-Bu\|^2]\\ &= \|x_n-u\|^2+(1-\alpha_n)r_n(r_n-2\alpha)\|Bx_n-Bu\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}(1-d)\gamma(2\alpha-\tau)\|Bx_n-Bu\|^2 &\le (1-\alpha_n)r_n(2\alpha-r_n)\|Bx_n-Bu\|^2\\ &\le \|x_n-u\|^2-\|x_{n+1}-u\|^2.\end{aligned}$$

Hence, we obtain $\|Bx_n-Bu\|\to 0$.

For $u\in\Omega$, we have, from Lemma 2.1,

$$\begin{aligned}\|u_n-u\|^2 &= \|T_{r_n}(x_n-r_nBx_n)-T_{r_n}(u-r_nBu)\|^2\\ &\le \langle T_{r_n}(x_n-r_nBx_n)-T_{r_n}(u-r_nBu), x_n-r_nBx_n-(u-r_nBu)\rangle\\ &= \frac{1}{2}\{\|u_n-u\|^2+\|x_n-r_nBx_n-(u-r_nBu)\|^2-\|x_n-r_nBx_n-(u-r_nBu)-(u_n-u)\|^2\}\\ &\le \frac{1}{2}\{\|u_n-u\|^2+\|x_n-u\|^2-\|x_n-r_nBx_n-(u-r_nBu)-(u_n-u)\|^2\}\\ &= \frac{1}{2}\{\|u_n-u\|^2+\|x_n-u\|^2-\|x_n-u_n\|^2+2r_n\langle Bx_n-Bu, x_n-u_n\rangle-r_n^2\|Bx_n-Bu\|^2\}.\end{aligned}$$

Hence,

$$\begin{aligned}\|u_n-u\|^2 &\le \|x_n-u\|^2-\|x_n-u_n\|^2+2r_n\langle Bx_n-Bu, x_n-u_n\rangle-r_n^2\|Bx_n-Bu\|^2\\ &\le \|x_n-u\|^2-\|x_n-u_n\|^2+2r_n\langle Bx_n-Bu, x_n-u_n\rangle.\end{aligned}$$

Then, by (3.3) and (3.2),

$$\begin{aligned}\|x_{n+1}-u\|^2 &\le \alpha_n\|x_n-u\|^2+(1-\alpha_n)\|t_n-u\|^2\\ &\le \alpha_n\|x_n-u\|^2+(1-\alpha_n)\|u_n-u\|^2\\ &\le \alpha_n\|x_n-u\|^2+(1-\alpha_n)[\|x_n-u\|^2-\|x_n-u_n\|^2+2r_n\langle Bx_n-Bu, x_n-u_n\rangle]\\ &\le \|x_n-u\|^2-(1-\alpha_n)\|x_n-u_n\|^2+(1-\alpha_n)2r_n\|Bx_n-Bu\|\|x_n-u_n\|.\end{aligned}$$

Hence,

$$\begin{aligned}(1-d)\|x_n-u_n\|^2 &\le (1-\alpha_n)\|x_n-u_n\|^2\\ &\le \|x_n-u\|^2-\|x_{n+1}-u\|^2+(1-\alpha_n)2r_n\|Bx_n-Bu\|\|x_n-u_n\|.\end{aligned}$$

Since $\|Bx_n-Bu\|\to 0$, $\{x_n\}$ and $\{u_n\}$ are bounded, we obtain $\|x_n-u_n\|\to 0$. From $\|t_n-x_n\|\le\|t_n-u_n\|+\|x_n-u_n\|$ we also have $\|t_n-x_n\|\to 0$.

For $u\in\Omega$, since $\|W_nt_n-u\|\le\|t_n-u\|\le\|x_n-u\|$, we have

$$\limsup_{n\to\infty}\|W_nt_n-u\|\le\theta.$$

Further, we have

$$\lim_{n\to\infty}\|\alpha_n(x_n-u)+(1-\alpha_n)(W_nt_n-u)\|=\lim_{n\to\infty}\|x_{n+1}-u\|=\theta.$$

By Lemma 2.2, we obtain $\lim_{n\to\infty}\|W_nt_n-x_n\|=0$.

Since $\|W_nx_n-x_n\|\le\|W_nx_n-W_nt_n\|+\|W_nt_n-x_n\|\le\|x_n-t_n\|+\|W_nt_n-x_n\|$, we have

$$\|W_nx_n-x_n\|\to 0. \tag{3.5}$$

At the same time, observe that

$$\|Wx_n-x_n\|\le\|Wx_n-W_nx_n\|+\|W_nx_n-x_n\|. \tag{3.6}$$

It follows from (3.5) and (3.6) and Remark 3.1, we have $\lim_{n\to\infty}\|Wx_n-x_n\|=0$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i}\rightharpoonup w$. From $\|x_n-u_n\|\to 0$, we obtain that $u_{n_i}\rightharpoonup w$. Since $\{u_{n_i}\}\subset C$ and C is closed and convex, we obtain $w\in C$.

First, we show $w \in GMEP(F, \varphi, B)$. By $u_n = T_{r_n}(x_n - r_n Bx_n)$, we know that

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

It follows from (A2) that

$$\varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n), \quad \forall y \in C.$$

Hence,

$$\varphi(y) - \varphi(u_{n_i}) + \langle Bx_{n_i}, y - u_{n_i}\rangle + \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \geq F(y, u_{n_i}), \quad \forall y \in C. \tag{3.7}$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we obtain $y_t \in C$. So, from (3.7) we have

$$\begin{aligned}\langle y_t - u_{n_i}, By_t\rangle &\geq \langle y_t - u_{n_i}, By_t\rangle - \varphi(y_t) + \varphi(u_{n_i}) - \langle y_t - u_{n_i}, Bx_{n_i}\rangle - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + F(y_t, u_{n_i}) \\ &= \langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle + \langle y_t - u_{n_i}, Bu_{n_i} - Bx_{n_i}\rangle - \varphi(y_t) + \varphi(u_{n_i}) \\ &\quad - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + F(y_t, u_{n_i}).\end{aligned}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|Bu_{n_i} - Bx_{n_i}\| \to 0$. Further, from the inverse strong monotonicity of B, we have $\langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle \geq 0$. So, from (A4), (A5), and the weak lower semicontinuity of φ, $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup w$, we have at the limit

$$\langle y_t - w, By_t\rangle \geq -\varphi(y_t) + \varphi(w) + F(y_t, w), \tag{3.8}$$

as $i \to \infty$. From (A1), (A4) and (3.8), we also get

$$\begin{aligned} 0 &= F(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\ &\leq tF(y_t, y) + (1-t)F(y_t, w) + t\varphi(y) + (1-t)\varphi(w) - \varphi(y_t) \\ &= t[F(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)[F(y_t, w) + \varphi(w) - \varphi(y_t)] \\ &\leq t[F(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)\langle y_t - w, By_t\rangle \\ &= t[F(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)t\langle y - w, By_t\rangle, \end{aligned}$$
$$0 \leq F(y_t, y) + \varphi(y) - \varphi(y_t) + (1-t)\langle y - w, By_t\rangle.$$

Letting $t \to 0$, we have, for each $y \in C$,

$$F(w, y) + \varphi(y) - \varphi(w) + \langle y - w, Bw\rangle \geq 0.$$

This implies that $w \in GMEP(F, \varphi, B)$.

We next show that $w \in Fix(W)$. Assume that $w \notin Fix(W)$. Since W is nonexpansive, $x_{n_i} \rightharpoonup w$ and $w \neq Ww$, from the Opial condition, we have

$$\begin{aligned}\liminf_{i\to\infty}\|x_{n_i} - w\| &< \liminf_{i\to\infty}\|x_{n_i} - Ww\| \\ &\leq \liminf_{i\to\infty}\{\|x_{n_i} - Wx_{n_i}\| + \|Wx_{n_i} - Ww\|\} \\ &\leq \liminf_{i\to\infty}\|x_{n_i} - w\|\end{aligned}$$

which is a contradiction. It follows from Lemma 3.2 that $w \in Fix(W) = \cap_{i=1}^{\infty} Fix(S_i)$.

Finally we show $w \in VI(C, A)$. Let $T : H \to 2^H$ be defined as the same as that in Section 2. We have already mentioned that in this case the mapping T is maximal monotone, and $0 \in Tv$ if and only if $v \in VI(C, A)$. Let $(v, g) \in G(T)$. Then $Tv = Av + N_C v$ and hence $g - Av \in N_C v$. So, we have $\langle v - t, g - Av\rangle \geq 0$ for all $t \in C$. On the other hand, from $t_n = P_C(u_n - \lambda_n Ay_n)$ and $v \in C$ we have

$$\langle u_n - \lambda_n Ay_n - t_n, t_n - v\rangle \geq 0$$

and hence

$$\left\langle v - t_n, \frac{t_n - u_n}{\lambda_n} + Ay_n\right\rangle \geq 0.$$

Therefore, we have

$$\begin{aligned}\langle v - t_{n_i}, g\rangle &\geq \langle v - t_{n_i}, Av\rangle \\ &\geq \langle v - t_{n_i}, Av\rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}} + Ay_{n_i}\right\rangle \\ &= \left\langle v - t_{n_i}, Av - Ay_{n_i} - \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}}\right\rangle \\ &= \left\langle v - t_{n_i}, Av - At_{n_i} + At_{n_i} - Ay_{n_i} - \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}}\right\rangle \\ &= \langle v - t_{n_i}, Av - At_{n_i}\rangle + \langle v - t_{n_i}, At_{n_i} - Ay_{n_i}\rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}}\right\rangle \\ &\geq \langle v - t_{n_i}, At_{n_i} - Ay_{n_i}\rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}}\right\rangle\end{aligned}$$

Hence we obtain $\langle v - w, g\rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $w \in T^{-1}0$ and hence $w \in VI(C, A)$. This implies $w \in \Omega$.

Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z$. Then $z \in \Omega$. Let us show that $w = z$. Assume that $w \neq z$. From the Opial condition, we have

$$\begin{aligned}\lim_{n\to\infty} \|x_n - w\| &= \liminf_{i\to\infty} \|x_{n_i} - w\| < \liminf_{i\to\infty} \|x_{n_i} - z\| \\ &= \lim_{n\to\infty} \|x_n - z\| = \liminf_{j\to\infty} \|x_{n_j} - z\| \\ &< \liminf_{j\to\infty} \|x_{n_j} - w\| = \lim_{n\to\infty} \|x_n - w\|.\end{aligned}$$

This is a contradiction. Thus, we have $w = z$. This implies that $x_n \rightharpoonup w \in \Omega$.

Now put $w_n = P_\Omega(x_n)$. We show that $w = \lim_{n\to\infty} w_n$.

From $w_n = P_\Omega(x_n)$ and $w \in \Omega$, we have

$$\langle w - w_n, w_n - x_n\rangle \geq 0.$$

From (3.3) and Lemma 2.3, we know that $\{w_n\}$ converges strongly to some $w_0 \in \Omega$. Then, we have

$$\langle w - w_0, w_0 - w\rangle \geq 0$$

and hence $w = w_0$. The proof is now complete. □

Remark 3.2. It follows from the proof of Theorem 3.1 that $\|x_n - u_n\| \to 0$ and $\|y_n - u_n\| \to 0$. Hence, we have also $u_n \rightharpoonup w$ and $y_n \rightharpoonup w$, where $w = \lim_{n\to\infty} P_\Omega x_n$.

By Theorem 3.1, we can obtain many new and interesting weak convergence theorems for some algorithms. Now we only give some examples as follows:

Let $A = 0$, by Theorem 3.1, we have the following result:

Corollary 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)–(A5) *and $\varphi : C \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let B be an α-inverse strongly monotone mapping from C into H. Let $S_1, S_2, \ldots$ be a family of infinitely nonexpansive mappings of C into itself such that $\Theta = \cap_{i=1}^{\infty} Fix(S_i) \cap GMEP(F, \varphi, B) \neq \emptyset$ and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i \leq \delta < 1$ for every $i \in N$. For any $n \in N$, let W_n be the W-mapping from C into itself generated by $S_n, S_{n-1}, \ldots, S_1$ and $\xi_n, \xi_{n-1}, \ldots, \xi_1$. Assume that either* (B1) *or* (B2) *holds. Let $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) W_n u_n, \end{cases}$$

for every $n = 1, 2, \ldots$. If $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$ and $\{r_n\} \subset [\gamma, \tau]$ for some $\gamma, \tau \in (0, 2\alpha)$, then, $\{x_n\}$ and $\{u_n\}$ converge weakly to $w \in \Theta$, where $w = \lim_{n\to\infty} P_\Theta(x_n)$.

If $\varphi = 0$ and $B = 0$, by Corollary 3.1, we have:

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)–(A5)*. Let $S_1, S_2, \ldots$ be a family of infinitely nonexpansive mappings of C into itself such that $\Delta = \cap_{i=1}^{\infty} Fix(S_i) \cap EP(F) \neq \emptyset$ and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i \leq \delta < 1$ for every $i \in N$. For any $n \in N$, let W_n be the W-mapping from C into itself generated by $S_n, S_{n-1}, \ldots, S_1$ and $\xi_n, \xi_{n-1}, \ldots, \xi_1$. Assume that either* (B1) *or* (B2) *holds. Let $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) W_n u_n, \end{cases}$$

for every $n = 1, 2, \ldots$. If $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$ and $\{r_n\} \subset [\gamma, +\infty)$ for some $\gamma > 0$, then, $\{x_n\}$ and $\{u_n\}$ converge weakly to $w \in \Delta$, where $w = \lim_{n\to\infty} P_{\Delta}(x_n)$.

Let $B = 0$, by Theorem 3.1, we obtain the following weak convergence theorems for an algorithm of finding solutions of problem (1.2):

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)–(A5) *and $\varphi : C \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be a monotone and K-Lipschitz continuous mapping from C into H. Let $S_1, S_2, \ldots$ be a family of infinitely nonexpansive mappings of C into itself such that $\Sigma = \cap_{i=1}^{\infty} Fix(S_i) \cap VI(C, A) \cap MEP(F, \varphi) \neq \emptyset$ and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i \leq \delta < 1$ for every $i \in N$. For any $n \in N$, let W_n be the W-mapping from C into itself generated by $S_n, S_{n-1}, \ldots, S_1$ and $\xi_n, \xi_{n-1}, \ldots, \xi_1$. Assume that either* (B1) *or* (B2) *holds. Let $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \varphi(y) - \varphi(u_n) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) W_n P_C(u_n - \lambda_n A y_n), \end{cases}$$

for every $n = 1, 2, \ldots$. If $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{K})$, $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$ and $\{r_n\} \subset [\gamma, +\infty)$ for some $\gamma > 0$, then, $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ converge weakly to $w \in \Sigma$, where $w = \lim_{n\to\infty} P_{\Sigma}(x_n)$.

If $F = 0$ and $\varphi = 0$, by Corollary 3.3, we have:

Corollary 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a monotone and K-Lipschitz continuous mapping from C into H. Let $S_1, S_2, \ldots$ be a family of infinitely nonexpansive mappings of C into itself such that $\Xi = \cap_{i=1}^{\infty} Fix(S_i) \cap VI(C, A) \neq \emptyset$ and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i \leq \delta < 1$ for every $i \in N$. For any $n \in N$, let W_n be the W-mapping from C into itself generated by $S_n, S_{n-1}, \ldots, S_1$ and $\xi_n, \xi_{n-1}, \ldots, \xi_1$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) W_n P_C(x_n - \lambda_n A y_n), \end{cases}$$

for every $n = 1, 2, \ldots$. If $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{K})$, $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$, then, $\{x_n\}$ and $\{y_n\}$ converge weakly to $w \in \Xi$, where $w = \lim_{n\to\infty} P_{\Xi}(x_n)$.

If $C = H$, then $P_H = I$ and $A^{-1}0 = VI(C, A)$, by Corollary 3.4, we have:

Corollary 3.5. *Let H be a real Hilbert space. Let A be a monotone and K-Lipschitz continuous mapping of H into itself. Let $S_1, S_2, \ldots$ be a family of infinitely nonexpansive mappings of H into itself such that $\Gamma = \cap_{i=1}^{\infty} Fix(S_i) \cap A^{-1}(0) \neq \emptyset$ and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i \leq \delta < 1$ for every $i \in N$. For any $n \in N$, let W_n be the W-mapping of H into itself generated by $S_n, S_{n-1}, \ldots, S_1$ and $\xi_n, \xi_{n-1}, \ldots, \xi_1$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) W_n P_C(x_n - \lambda_n A(x_n - \lambda_n A x_n)), \end{cases}$$

for every $n = 1, 2, \ldots$. If $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{K})$, $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$, then, $\{x_n\}$ and $\{y_n\}$ converge weakly to $w \in \Gamma$, where $w = \lim_{n\to\infty} P_{\Gamma}(x_n)$.

Now we prove a weak convergence theorem of a new iterative process for finding a common element of the set of solutions of a generalized mixed equilibrium problem, the set of fixed points of a family of infinitely many nonexpansive mappings and the set of fixed points of a Lipschitz pseudocontractive mapping.

Corollary 3.6. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)–(A5) *and $\varphi : C \to R\cup\{+\infty\}$ be a proper lower semicontinuous and convex function. Let T be a pseudocontractive and m-Lipschitz continuous mapping from C into itself and B be an α-inverse strongly monotone mapping from C into H. Let $S_1, S_2, \ldots$ be a family of infinitely nonexpansive mappings of C into itself such that $\Pi = \cap_{i=1}^{\infty} Fix(S_i) \cap Fix(T) \cap GMEP(F, \varphi, B) \neq \emptyset$ and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i \leq \delta < 1$ for every $i \in N$. For any $n \in N$, let W_n be the W-mapping from C into itself generated by $S_n, S_{n-1}, \ldots, S_1$ and $\xi_n, \xi_{n-1}, \ldots, \xi_1$. Assume that either* (B1) *or* (B2) *holds. Let $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n(u_n - Tu_n)), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)W_n P_C(u_n - \lambda_n(y_n - Ty_n)), \end{cases}$$

for every $n = 1, 2, \ldots$. If $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{m+1})$, $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$ and $\{r_n\} \subset [\gamma, \tau]$ for some $\gamma, \tau \in (0, 2\alpha)$, then, $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ converge weakly to $w \in \Pi$, where $w = \lim_{n\to\infty} P_\Pi(x_n)$.

Proof. Let $A = I - T$. Let us show that the mapping A is monotone and $(m + 1)$-Lipschitz continuous. From the definition of the mapping A, we have

$$\begin{aligned} \langle Ax - Ay, x - y\rangle &= \langle x - y - Tx + Ty, x - y\rangle \\ &= \|x - y\|^2 - \langle Tx - Ty, x - y\rangle \geq \|x - y\|^2 - \|x - y\|^2 = 0. \end{aligned}$$

So, A is monotone. We also have

$$\begin{aligned} \|Ax - Ay\|^2 &= \|(I - T)x - (I - T)y\| \\ &= \|x - y\|^2 + \|Tx - Ty\|^2 - 2\langle x - y, Tx - Ty\rangle \\ &\leq \|x - y\|^2 + m\|x - y\|^2 + 2\|x - y\|\|Tx - Ty\| \\ &\leq \|x - y\|^2 + m\|x - y\|^2 + 2m\|x - y\|^2 = (m + 1)^2\|x - y\|^2. \end{aligned}$$

So, we have $\|Ax - Ay\| \leq (m + 1)\|x - y\|$ and A is $(m + 1)$-Lipschitz continuous. It is easy to check that $Fix(T) = VI(C, A)$. By Theorem 3.1 we obtain the desired result. □

4. The case of a finite family of nonexpansive mappings

In this section, we show a weak convergence theorem of an iterative algorithm based on extragradient method which solves the problem of finding a common element of the set of solutions of a generalized mixed equilibrium problem, the set of fixed points of a finite family of nonexpansive mappings and the set of solutions of a variational inequality problem for a monotone, Lipschitz continuous mapping in a Hilbert space.

Let $\{T_i\}_{i=1}^{N}$ be a family of finitely many nonexpansive mappings of C into itself and let $\zeta_1, \zeta_2, \ldots, \zeta_N$ be real numbers such that $0 \leq \zeta_i \leq 1$ for every $i = 1, 2, \ldots, N$. We define a mapping $\widetilde{W}$ of C into itself as follows

$$\begin{aligned} &U_1 = \zeta_1 T_1 + (1 - \zeta_1)I, \\ &U_2 = \zeta_2 T_2 U_1 + (1 - \zeta_2)I, \\ &\cdots \\ &U_{N-1} = \zeta_{N-1} T_{N-1} U_{N-2} + (1 - \zeta_{N-1})I, \\ &\widetilde{W} := U_N = \zeta_N T_N U_{N-1} + (1 - \zeta_N)I. \end{aligned}$$

Such a mapping $\widetilde{W}$ is called the $\widetilde{W}$-mapping generated by $T_1, T_2, \ldots, T_N$ and $\zeta_1, \zeta_2, \ldots, \zeta_N$.

This concept of $\widetilde{W}$-mappings was introduced in [30,31]. It is now one of the main tools in studying convergence of iterative methods for approaching a common fixed points of nonlinear mappings; more recent progresses can be found in [12,32,33] and the references cited therein.

Lemma 4.1 (*see [12]*). *Let C be a nonempty convex subset of a Banach space. Let $\{T_i\}_{i=1}^{N}$ be a family of finitely nonexpansive mappings of C into itself and $\{\zeta_{n,1}\}, \{\zeta_{n,2}\}, \ldots, \{\zeta_{n,N}\}$ be sequences in* $[0, 1]$ *such that $\zeta_{n,i} \to \zeta_i$ $(i = 1, \ldots, N)$. Moreover for every integer $n \geq 1$, let $\widetilde{W}$ and $\widetilde{W}_n$ be the $\widetilde{W}$-mappings generated by $T_1, T_2, \ldots, T_N$ and $\zeta_1, \zeta_2, \ldots, \zeta_N$ and $T_1, T_2, \ldots, T_N$ and $\{\zeta_{n,1}\}\{\zeta_{n,2}\}, \ldots, \{\zeta_{n,N}\}$, respectively. Then for every $x \in C$, it follows that*

$$\lim_{n\to\infty} \|\widetilde{W}_n x - \widetilde{W}x\| = 0.$$

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)–(A5) *and $\varphi : C \to R\cup\{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be a monotone and K-Lipschitz continuous mapping from C into H and B be an α-inverse strongly monotone mapping from C into H. Let $T_1, T_2, \ldots, T_N$*

be a family of finitely nonexpansive mappings of C into itself such that $\Omega_1 = \bigcap_{n=1}^{N} Fix(T_i) \cap VI(C, A) \cap GMEP(F, \varphi, B) \neq \emptyset$. Let $\{\zeta_{n,1}\}, \{\zeta_{n,2}\}, \ldots, \{\zeta_{n,N}\}$ be sequences in $[\varepsilon_1, \varepsilon_2]$ with $0 < \varepsilon_1 \leq \varepsilon_2 < 1$. Let $\widetilde{W}_n$ be the $\widetilde{W}$-mapping generated by $T_1, T_2, \ldots, T_N$ and $\zeta_{n,1}, \zeta_{n,2}, \ldots, \zeta_{n,N}$. Assume that either (B1) *or* (B2) *holds. Let $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)\widetilde{W}_n P_C(u_n - \lambda_n A y_n), \end{cases}$$

for every $n = 1, 2, \ldots$. If $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{K})$, $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$ and $\{r_n\} \subset [\gamma, \tau]$ for some $\gamma, \tau \in (0, 2\alpha)$, then, $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ converge weakly to $w \in \Omega_1$, where $w = \lim_{n\to\infty} P_{\Omega_1}(x_n)$.

Proof. Put $t_n = P_C(u_n - \lambda_n A y_n)$ for every $n = 1, 2, \ldots$. Let $u \in \Omega$ and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.1. Then $u = P_C(u - \lambda_n A u) = T_{r_n}(u - r_n B u)$. From $u_n = T_{r_n}(x_n - r_n B x_n) \in C$ and the proof of Theorem 3.1, we have

$$\|u_n - u\|^2 \leq \|x_n - u\|^2 + r_n(r_n - 2\alpha)\|Bx_n - Bu\|^2 \leq \|x_n - u\|, \tag{4.1}$$

$$\|t_n - u\|^2 \leq \|u_n - u\|^2 + ({\lambda_n}^2K^2 - 1)\|u_n - y_n\|^2 \leq \|u_n - u\|^2, \tag{4.2}$$

$$\|t_n - u\|^2 \leq \|u_n - u\|^2 + ({\lambda_n}^2K^2 - 1)\|y_n - t_n\|^2 \leq \|u_n - u\|^2, \tag{4.3}$$

and

$$\|u_n - u\|^2 \leq \|x_n - u\|^2 - \|x_n - u_n\|^2 + 2r_n\langle Bx_n - Bu, x_n - u_n\rangle. \tag{4.4}$$

By Lemma 3.1 in [30], we know that $\widetilde{W}_n$ is nonexpansive and $Fix(\widetilde{W}_n) = \bigcap_{i=1}^{N} Fix(T_i)$. It follows from (4.1), (4.2), $x_{n+1} = \alpha_n x_n + (1-\alpha_n)\widetilde{W}_n t_n$ and $u = \widetilde{W}_n u$ that

$$\begin{aligned} \|x_{n+1} - u\|^2 &= \|\alpha_n x_n + (1-\alpha_n)\widetilde{W}_n t_n - u\|^2 \\ &\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|\widetilde{W}_n t_n - u\|^2 \\ &\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|t_n - u\|^2 \\ &\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)[\|u_n - u\|^2 + ({\lambda_n}^2K^2 - 1)\|u_n - y_n\|^2] \\ &\leq \|x_n - u\|^2 + (1-\alpha_n)({\lambda_n}^2K^2 - 1)\|u_n - y_n\|^2 \\ &\leq \|x_n - u\|^2, \end{aligned} \tag{4.5}$$

for every $n = 1, 2, \ldots$. Therefore, there exists $\theta = \lim_{n\to\infty}\|x_n - u\|$ and $\{x_n\}$ is bounded. From (4.1) and (4.2), we also obtain that $\{t_n\}$ and $\{u_n\}$ are bounded.

By (4.5), we have

$$\|u_n - y_n\|^2 \leq \frac{1}{(1-\alpha_n)(1-{\lambda_n}^2K^2)}\left(\|x_n - u\|^2 - \|x_{n+1} - u\|^2\right).$$

Hence, $\|u_n - y_n\| \to 0$.

It follows from (4.5) and (4.3) that

$$\begin{aligned} \|x_{n+1} - u\|^2 &\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|t_n - u\|^2 \\ &\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)[\|u_n - u\|^2 + ({\lambda_n}^2K^2 - 1)\|y_n - t_n\|^2] \\ &\leq \|x_n - u\|^2 + (1-\alpha_n)({\lambda_n}^2K^2 - 1)\|y_n - t_n\|^2 \\ &\leq \|x_n - u\|^2. \end{aligned}$$

And thus

$$\|t_n - y_n\|^2 \leq \frac{1}{(1-\alpha_n)(1-{\lambda_n}^2K^2)}\left(\|x_n - u\|^2 - \|x_{n+1} - u\|^2\right).$$

So, we have $\|t_n - y_n\| \to 0$. From $\|u_n - t_n\| \leq \|u_n - y_n\| + \|y_n - t_n\|$ we also have $\|u_n - t_n\| \to 0$. As A is K-Lipschitz continuous, we have $\|Ay_n - At_n\| \to 0$.

Using similar argument as in the proof of Theorem 3.1, by (4.5), (4.4), (4.1) and (4.2), we can obtain $\|Bx_n - Bu\| \to 0$ and $\|x_n - u_n\| \to 0$. From $\|t_n - x_n\| \leq \|t_n - u_n\| + \|x_n - u_n\|$ we also have $\|t_n - x_n\| \to 0$.

For $u \in \Omega_1$, since $\|\widetilde{W}_n t_n - u\| \leq \|t_n - u\| \leq \|x_n - u\|$, we have

$$\limsup_{n\to\infty}\|\widetilde{W}_n t_n - u\| \leq \theta.$$

Further, we have

$$\lim_{n\to\infty} \|\alpha_n(x_n - u) + (1-\alpha_n)(\widetilde{W}_n t_n - u)\| = \lim_{n\to\infty} \|x_{n+1} - u\| = \theta.$$

By Lemma 2.2, we obtain $\lim_{n\to\infty} \|\widetilde{W}_n t_n - x_n\| = 0$.

Since $\|\widetilde{W}_n x_n - x_n\| \le \|\widetilde{W}_n x_n - \widetilde{W}_n t_n\| + \|\widetilde{W}_n t_n - x_n\| \le \|x_n - t_n\| + \|\widetilde{W}_n t_n - x_n\|$, we have

$$\|\widetilde{W}_n x_n - x_n\| \to 0. \tag{4.6}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w$. From $\|x_n - u_n\| \to 0$, we obtain that $u_{n_i} \rightharpoonup w$. Since $\{u_{n_i}\} \subset C$ and C is closed and convex, we obtain $w \in C$.

In order to show that $w \in \Omega_1$, we first show that $w \in \cap_{i=1}^N Fix(T_i)$. To see this, we observe that we may assume (by passing to a further subsequence if necessary) that $\zeta_{n_i,k} \to \zeta_k$ for $k = 1, 2, \ldots, N$. Let $\widetilde{W}$ be the $\widetilde{W}$-mapping generated by $T_1, T_2, \ldots, T_N$ and $\zeta_1, \zeta_2, \ldots, \zeta_N$. By Lemma 3.1 in [30], we know that $\widetilde{W}$ is nonexpansive and $\cap_{i=1}^N Fix(T_i) = Fix(\widetilde{W})$. It follows from Lemma 4.1 that

$$\widetilde{W}_{n_i} x \to \widetilde{W} x, \quad \text{for all } x \in C. \tag{4.7}$$

Assume that $w \notin Fix(\widetilde{W})$. Since $x_{n_i} \rightharpoonup w$ and $w \neq \widetilde{W} w$, it follows from the Opial condition, (4.6) and (4.7) that

$$\begin{aligned}
\liminf_{i\to\infty} \|x_{n_i} - w\| &< \liminf_{i\to\infty} \|x_{n_i} - \widetilde{W} w\| \\
&\le \liminf_{i\to\infty} \{\|x_{n_i} - \widetilde{W}_{n_i} x_{n_i}\| + \|\widetilde{W}_{n_i} x_{n_i} - \widetilde{W} x_{n_i}\| + \|\widetilde{W} x_{n_i} - \widetilde{W} w\|\} \\
&\le \liminf_{i\to\infty} \|x_{n_i} - w\|
\end{aligned}$$

which is a contradiction. Hence, we have $w \in Fix(W) = \cap_{i=1}^N Fix(T_i)$.

From the proof of Theorem 3.1, it is easy to see that $w \in GMEP(F, \varphi, B)$ and $w \in VI(C, A)$. Thus, we have $w \in \Omega_1$. The rest of the proof is the same as that in the proof of Theorem 3.1. The proof is now complete. □

Using similar arguments in Section 3, by Theorem 4.1, we also can obtain the following weak convergence theorems for some algorithms.

Corollary 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)–(A5) *and $\varphi : C \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let B be an α-inverse strongly monotone mapping from C into H. Let $T_1, T_2, \ldots, T_N$ be a family of finitely nonexpansive mappings of C into itself such that $\Theta_1 = \cap_{n=1}^N Fix(T_i) \cap GMEP(F, \varphi, B) \neq \emptyset$. Let $\{\zeta_{n,1}\}, \{\zeta_{n,2}\}, \ldots, \{\zeta_{n,N}\}$ be sequences in $[\varepsilon_1, \varepsilon_2]$ with $0 < \varepsilon_1 \le \varepsilon_2 < 1$. Let $\widetilde{W}_n$ be the $\widetilde{W}$-mapping generated by $T_1, T_2, \ldots, T_N$ and $\zeta_{n,1}, \zeta_{n,2}, \ldots, \zeta_{n,N}$. Assume that either* (B1) *or* (B2) *holds. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)\widetilde{W}_n u_n, \end{cases}$$

for every $n = 1, 2, \ldots$. If $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$ and $\{r_n\} \subset [\gamma, \tau]$ for some $\gamma, \tau \in (0, 2\alpha)$, then, $\{x_n\}$ and $\{u_n\}$ converge weakly to $w \in \Theta_1$, where $w = \lim_{n\to\infty} P_{\Theta_1}(x_n)$.

Corollary 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)–(A5). *Let $T_1, T_2, \ldots, T_N$ be a family of finitely nonexpansive mappings of C into itself such that $\Delta_1 = \cap_{n=1}^N Fix(T_i) \cap EP(F) \neq \emptyset$. Let $\{\zeta_{n,1}\}, \{\zeta_{n,2}\}, \ldots, \{\zeta_{n,N}\}$ be sequences in $[\varepsilon_1, \varepsilon_2]$ with $0 < \varepsilon_1 \le \varepsilon_2 < 1$. Let $\widetilde{W}_n$ be the $\widetilde{W}$-mapping generated by $T_1, T_2, \ldots, T_N$ and $\zeta_{n,1}, \zeta_{n,2}, \ldots, \zeta_{n,N}$. Assume that either* (B3) *or* (B2) *holds. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)\widetilde{W}_n u_n, \end{cases}$$

for every $n = 1, 2, \ldots$. If $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$ and $\{r_n\} \subset [\gamma, +\infty)$ for some $\gamma > 0$, then, $\{x_n\}$ and $\{u_n\}$ converge weakly to $w \in \Delta_1$, where $w = \lim_{n\to\infty} P_{\Delta_1}(x_n)$.

Corollary 4.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)–(A5) *and $\varphi : C \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be a monotone and K-Lipschitz continuous mapping from C into H. Let $T_1, T_2, \ldots, T_N$ be a family of finitely nonexpansive mappings of C into itself such that $\Sigma_1 = \cap_{n=1}^{N} Fix(T_i) \cap VI(C, A) \cap MEP(F, \varphi) \neq \emptyset$. Let $\{\zeta_{n,1}\}, \{\zeta_{n,2}\}, \ldots, \{\zeta_{n,N}\}$ be sequences in $[\varepsilon_1, \varepsilon_2]$ with $0 < \varepsilon_1 \leq \varepsilon_2 < 1$. Let $\widetilde{W}_n$ be the $\widetilde{W}$-mapping generated by $T_1, T_2, \ldots, T_N$ and $\zeta_{n,1}, \zeta_{n,2}, \ldots, \zeta_{n,N}$. Assume that either* (B1) *or* (B2) *holds. Let $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \varphi(y) - \varphi(u_n) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\widetilde{W}_n P_C(u_n - \lambda_n A y_n), \end{cases}$$

for every $n = 1, 2, \ldots$. If $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{K})$, $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$ and $\{r_n\} \subset [\gamma, +\infty)$ for some $\gamma > 0$, then, $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ converge weakly to $w \in \Sigma_1$, where $w = \lim_{n\to\infty} P_{\Sigma_1}(x_n)$.

Corollary 4.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a monotone and K-Lipschitz continuous mapping from C into H. Let $T_1, T_2, \ldots, T_N$ be a family of finitely nonexpansive mappings of C into itself such that $\Xi_1 = \cap_{n=1}^{N} Fix(T_i) \cap VI(C, A) \neq \emptyset$. Let $\{\zeta_{n,1}\}, \{\zeta_{n,2}\}, \ldots, \{\zeta_{n,N}\}$ be sequences in $[\varepsilon_1, \varepsilon_2]$ with $0 < \varepsilon_1 \leq \varepsilon_2 < 1$. Let $\widetilde{W}_n$ be the $\widetilde{W}$-mapping generated by $T_1, T_2, \ldots, T_N$ and $\zeta_{n,1}, \zeta_{n,2}, \ldots, \zeta_{n,N}$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\widetilde{W}_n P_C(x_n - \lambda_n A y_n), \end{cases}$$

for every $n = 1, 2, \ldots$. If $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{K})$, $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$, then, $\{x_n\}$ and $\{y_n\}$ converge weakly to $w \in \Xi_1$, where $w = \lim_{n\to\infty} P_{\Xi_1}(x_n)$.

Corollary 4.5. *Let H be a real Hilbert space. Let A be a monotone and K-Lipschitz continuous mapping of H into itself. Let $T_1, T_2, \ldots, T_N$ be a family of finitely nonexpansive mappings of C into itself such that $\Gamma_1 = \cap_{n=1}^{N} Fix(T_i) \cap A^{-1}(0) \neq \emptyset$. Let $\{\zeta_{n,1}\}, \{\zeta_{n,2}\}, \ldots, \{\zeta_{n,N}\}$ be sequences in $[\varepsilon_1, \varepsilon_2]$ with $0 < \varepsilon_1 \leq \varepsilon_2 < 1$. Let $\widetilde{W}_n$ be the $\widetilde{W}$-mapping generated by $T_1, T_2, \ldots, T_N$ and $\zeta_{n,1}, \zeta_{n,2}, \ldots, \zeta_{n,N}$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\widetilde{W}_n P_C(x_n - \lambda_n A(x_n - \lambda_n A x_n)), \end{cases}$$

for every $n = 1, 2, \ldots$. If $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{K})$, $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$, then, $\{x_n\}$ and $\{y_n\}$ converge weakly to $w \in \Gamma_1$, where $w = \lim_{n\to\infty} P_{\Gamma_1}(x_n)$.

Corollary 4.6. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)–(A5) *and $\varphi : C \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let T be a pseudocontractive and m-Lipschitz continuous mapping from C into itself and B be an α-inverse strongly monotone mapping from C into H. Let $T_1, T_2, \ldots, T_N$ be a family of finitely nonexpansive mappings of C into itself such that $\Pi_1 = \cap_{n=1}^{N} Fix(T_i) \cap Fix(T) \cap GMEP(F, \varphi, B) \neq \emptyset$. Let $\{\zeta_{n,1}\}, \{\zeta_{n,2}\}, \ldots, \{\zeta_{n,N}\}$ be sequences in $[\varepsilon_1, \varepsilon_2]$ with $0 < \varepsilon_1 \leq \varepsilon_2 < 1$. Let $\widetilde{W}_n$ be the $\widetilde{W}$-mapping generated by $T_1, T_2, \ldots, T_N$ and $\zeta_{n,1}, \zeta_{n,2}, \ldots, \zeta_{n,N}$. Assume that either* (B1) *or* (B2) *holds. Let $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n(u_n - Tu_n)), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\widetilde{W}_n P_C(u_n - \lambda_n(y_n - Ty_n)), \end{cases}$$

for every $n = 1, 2, \ldots$. If $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{m+1})$, $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$ and $\{r_n\} \subset [\gamma, \tau]$ for some $\gamma, \tau \in (0, 2\alpha)$, then, $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ converge weakly to $w \in \Pi_1$, where $w = \lim_{n\to\infty} P_{\Pi_1}(x_n)$.

Remark 4.1. (i) It is clear that Theorems 3.1 and 4.1, Corollaries 3.1–3.3 and 4.1–4.3 are all generalizations and extensions of Theorem 4.1 in [11].

(ii) Let $\widetilde{W}_n$ be replaced by a nonexpansive mapping S, by Corollaries 4.4 and 4.5, respectively, we recover Theorems 3.1 and 4.1 in [20].

References

[1] J.W. Peng, J.C. Yao, A New Hybrid-extragradient Method For Generalized Mixed Equilibrium Problems and Fixed Point Problems and variational inequality problems, Taiwan. J. Math. 12 (6) (2008) 1401–1432.
[2] L.C. Ceng, J.C. Yao, A hybrid iterative scheme for mixed equilibrium problems and fixed point problems, J. Comput. Appl. Math. 214 (2008) 186–201.
[3] J.W. Peng, J.C. Yao, Strong convergence theorems of iterative scheme based on the extragradient method for mixed equilibrium problems and fixed point problems, Math. Comput. Model. 49 (2009) 1816–1828.
[4] J.W. Peng, J.C. Yao, An iterative algorithm combining viscosity method with parallel method for a generalized equilibrium problem and strict pseudocontractions, Fixed Point Theory Appl. volume 2009, Article ID 794178, 21 pages.
[5] S. Takahashi, W. Takahashi, Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space, Nonlinear Anal. 69 (2008) 1025–1033.
[6] S.D. Flam, A.S. Antipin, Equilibrium programming using proximal-like algorithms, Math. Program. 78 (1997) 29–41.
[7] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Stud. 63 (1994) 123–145.
[8] P.L. Combettes, S.A. Hirstoaga, Equilibrium programming in Hilbert spaces, J. Nonlinear Convex Anal. 6 (2005) 117–136.
[9] S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 331 (2006) 506–515.
[10] Y. Su, M. Shang, X. Qin, An iterative method of solutions for equilibrium and optimization problems, Nonlinear Anal. 69 (2008) 2709–2719.
[11] A. Tada, W. Takahashi, Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem, J. Optim. Theory Appl. 133 (2007) 359–370.
[12] V. Colao, G. Marino, H.-K. Xu, An iterative method for finding common solutions of equilibrium and fixed point problems, J. Math. Anal. Appl. 344 (2008) 340–352.
[13] S. Plubtieng, R. Punpaeng, A new iterative method for equilibrium problems and fixed point problems of nonexpansive mappings and monotone mappings, Appl. Math. Comput. 197 (2008) 548–558.
[14] G.M. Korpelevich, The extragradient method for finding saddle points and other problems, Matecon 12 (1976) 747–756.
[15] B.S. He, Z.H. Yang, X.M. Yuan, An approximate proximal-extragradient type method for monotone variational inequalities, J. Math. Anal. Appl. 300 (2004) 362–374.
[16] R. Gárciga Otero, A. Iuzem, Proximal methods with penalization effects in Banach spaces, Numer. Funct. Anal. Optim. 25 (2004) 69–91.
[17] M.V. Solodov, B.F. Svaiter, An inexact hybrid generalized proximal point algorithm and some new results on the theory of Bregman functions, Math. Oper. Res. 25 (2000) 214–230.
[18] M.V. Solodov, Convergence rate analysis of interactive algorithms for solving variational inequality problem, Math. Program. 96 (2003) 513–528.
[19] L.C. Zeng, J.C. Yao, Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems, Taiwan. J. Math. 10 (2006) 1293–1303.
[20] N. Nadezhkina, W. Takahashi, Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings, J. Optim. Theory Appl. 128 (2006) 191–201.
[21] Y. Yao, J.C. Yao, On modified iterative method for nonexpansive mappings and monotone mappings, Appl. Math. Comput. 186 (2) (2007) 1551–1558.
[22] Z. Opial, Weak convergence of the sequence of successive approximation for nonexpansive mappings, Bull. Amer. Math. Soc.. 73 (1967) 561–597.
[23] K. Goebel, W.A. Kirk, Topics on Metric Fixed-Point Theory, Cambridge University Press, Cambridge, England, 1990.
[24] F.E. Browder, W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, J. Math. Anal. Appl. 20 (1967) 197–228.
[25] R.T. Rockafellar, On the maximality of sums of nonlinear monotone operators, Trans. Amer. Math. Soc.. 149 (1970) 75–88.
[26] R.T. Rockafellar, Monotone operators and the proximal point algorithm, SIAM J. Control Optim. 14 (5) (1976) 877–898.
[27] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull. Austral. Math. Soc. 43 (1991) 153–159.
[28] W. Takahashi, M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings, J. Optim. Theory Appl. 118 (2003) 417–428.
[29] K. Shimoji, W. Takahashi, Strong convergence to common fixed points of infinite nonexpansive mappings and applications, Taiwan. J. Math. 5 (2001) 387–404.
[30] W. Takahashi, K. Shimoji, Convergence theorems for nonexpansive mappings and feasibility problems, Math. Comput. Model. 32 (2000) 1463–1471.
[31] W. Takahashi, Weak and strong convergence theorems for families of nonexpansive mappings and their applications, Ann. Univ. Mariae Curie–Sklodowska 51 (1997) 277–292.
[32] S. Atsushiba, W. Takahashi, Strong convergence theorems for a finite family of nonexpansive mappings and applications, Indian J. Math. 41 (1999) 435–453.
[33] L.C. Ceng, P. Cubiotti, J.C. Yao, Strong convergence theorems for finitely many nonexpansive mappings and applications, Nonlinear Anal. 67 (2007) 1464–1473.